UNIFORMLY VALID APPROXIMATIONS IN TWO-DIMENSIONAL SUBSONIC THIN AIRFOIL THEORY

by

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1. Introduction

Classical thin airfoil theory breaks down near stagnation points and near the edges of the airfoil. This is due to the assumption of the existence of a small perturbation flow field that is not valid in the neighbourhood of such points. For a survey of these difficulties and the methods that are developed to relieve them we refer to M. van Dyke [1].

In 1949 M.J. Lighthill [2] introduced a method to give uniformly valid approximate solutions of physical problems containing a small parameter ϵ. He demonstrated his method with a slightly non-linear first order ordinary differential equation and obtained a uniform approximation by taking both the dependent and the independent variable as a function of a new variable and then expanding both in a perturbation power series in ϵ. The method appeared to be very successful in the case of hyperbolic partial differential equations, because the characteristic variables could be used as Lighthill-variables in a natural way. (H.S. Tsien [3]).

An application of the method to equations of the elliptical type was given by Lighthill [4] in 1951. He constructed a uniformly valid approximation of the two-dimensional, incompressible thin airfoil problem by slightly shifting the flow field parallel to the chord of the airfoil, thus moving the troublesome singularity at the leading edge into the interior of the airfoil-contour. It appeared impossible to give a uniformly valid solution to any order of approximation in this way.

In this paper the thin airfoil problem is treated analogously, but the coordinate straining is more systematic and general: instead of a mere shifting of coordinates it concerns a function approximating uniformly a mapping of the physical plane onto a plane in which the airfoil is represented by its chord. It will appear in the following that uniformly valid approximations to any order can be obtained for airfoils with elliptically blunt and wedge-shaped sharp edges.

The main part of the paper is devoted to the incompressible case. In section 7 the method is considered from a more general point of view in order to extend the treatment to the compressible case.

2. Formulation of the problem

A two-dimensional incompressible nonviscous fluid flow in a x, y-plane is completely determined by a complex velocity potential χ, which is an analytic function of the complex variable $z = x + iy$:

$$\chi(z) = \Phi(x, y) + i\Psi(x, y). \tag{2.1}$$

The complex velocity $w(z) = u - iv$ can be found from

$$w(z) = \chi'(z) = u(x, y) - iv(x, y). \tag{2.2}$$

The equations 2.1 and 2.2 can be derived from (and are equivalent with) the equation of continuity

$$u_x + v_y = 0$$

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and the condition of irrotationality of the flow:

$$u_y - v_x = 0.$$

We will consider blunt bodies with small thickness parameter ϵ, of which the boundary is given by:

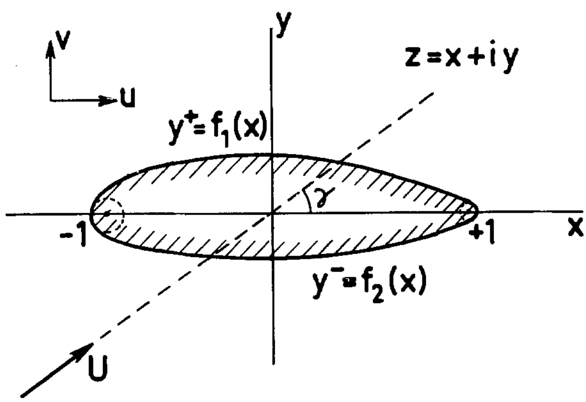
$$\begin{aligned} y^+(x) &= \epsilon f_1(x) = \epsilon(1-x^2)^{\frac{1}{2}} F_1(x), \\ y^-(x) &= \epsilon f_2(x) = \epsilon(1-x^2)^{\frac{1}{2}} F_2(x), \quad |x| \leqq 1, \end{aligned} \tag{2.3}$$

with $F_1(x)$ and $F_2(x)$ continuous, differentiable functions of x on the interval $|x| \leqq 1$, satisfying the condition

$$F_1(x) = -F_2(x) \neq 0 \text{ at } x = \pm 1.$$

So the thin airfoils under consideration have a finite, non-zero radius of curvature at the ends $x = \pm 1$. (See fig. 1)

Fig. 1.



The body is placed in a uniform flow with complex velocity-potential

$$\chi_H(z) = Ue^{-i\gamma} z, \tag{2.4}$$

of which the angle of incidence γ is not necessarily small.

The disturbance created by the presence of the body has to be zero at infinity, so for the total velocity potential $\chi(z)$ must hold:

$$\chi(z) \sim \chi_H(z) \text{ for } x^2+y^2 \to \infty \tag{2.5}$$

On the boundary of the body $\chi'(z)$ and $\chi(z)$ have to be finite, and because of the tangency of the flow, we have on the body

$$\Psi = \mathrm{Im}\ \chi = 0.$$

3. *The method of linearizing*

To obtain a uniformly valid approximate solution of the problem stated in section 2, we consider both the complex potential χ and the complex variable z as a function of a new complex variable $\eta = \alpha + i\beta$ and expand both in a perturbation power series in ϵ:

$$\chi(\eta) = \chi_0(\eta) + \epsilon\chi_1(\eta) + \epsilon^2\chi_2(\eta) + \ldots, \tag{3.1}$$

$$z(\eta) = \eta + \epsilon z_1(\eta) + \epsilon^2 z_2(\eta) + \ldots\ . \tag{3.2}$$

The functions χ_o, χ_1, $\chi_2, \ldots$ and z_1, $z_2, \ldots$ are analytic functions of the complex variable η. Separation into real and imaginary parts of series 3.1 and 3.2 gives:

$$\Phi(\alpha,\beta) = \Phi_o(\alpha,\beta) + \epsilon\Phi_1(\alpha,\beta) + \epsilon^2\Phi_2(\alpha,\beta) + \ldots \tag{3.3}$$

$$\Psi(\alpha,\beta) = \Psi_o(\alpha,\beta) + \epsilon\Psi_1(\alpha,\beta) + \epsilon^2\Psi_2(\alpha,\beta) + \ldots \tag{3.4}$$

$$x(\alpha,\beta) = \alpha + \epsilon x_1(\alpha,\beta) + \epsilon^2 x_2(\alpha,\beta) + \ldots \tag{3.5}$$

$$y(\alpha,\beta) = \beta + \epsilon y_1(\alpha,\beta) + \epsilon^2 y_2(\alpha,\beta) + \ldots \tag{3.6}$$

Taking z as a function of a new complex variable η means that the z-plane is mapped conformally on the η-plane. So the upper boundary of the airfoil transforms into

$$\begin{aligned}\beta + \epsilon y_1(\alpha,\beta) + \ldots &= \epsilon f_1\{\alpha + \epsilon x_1(\alpha,\beta) + \ldots\} = \\ &= \epsilon f_1(\alpha) + \epsilon^2 x_1(\alpha,\beta) f_1'(\alpha) + \epsilon^3(\ldots) + \ldots\end{aligned} \tag{3.7}$$

To the lowest order of approximation ($\epsilon = o$) this expression corresponds to $\beta = +o$, $|\alpha| \leqq 1$. It is easily seen that to the lowest order of approximation the lower side of the body transforms into $\beta = -o$, $|\alpha| \leqq 1$, that the edges $z = \pm 1$ are mapped onto $\eta = \pm 1$, and that the point at infinity remains undisturbed.

We now make use of the freedom that we have created by introducing the additional functions $z_1(\eta)$, $z_2(\eta), \ldots$:

We determine the analytic functions $z_1(\eta)$, $z_2(\eta), \ldots$ in such a way that to all orders of approximation the body is mapped on the line $\beta = \pm o$, $|\alpha| \leqq 1$ (with the edges $z = \pm 1$ corresponding to $\eta = \pm 1$). Furthermore we want the points at infinity to correspond apart from an unknown real factor $a(\epsilon) = 1 + a_1\epsilon + a_2\epsilon^2 + \ldots$:

$$z(\eta) \sim a(\epsilon)\eta = \eta + \epsilon a_1\eta + \epsilon^2 a_2\eta + \ldots \quad \text{for } \alpha^2 + \beta^2 \to \infty \tag{3.8}$$

Hence we want eq. 3.7 to be equivalent with $\beta = +o$, $|\alpha| \leqq 1$. So the coefficients of equal powers of ϵ must cancel on $\beta = +o$, $|\alpha| \leqq 1$:

$$\begin{aligned}&y_1(\alpha, +o) = f_1(\alpha),\\ &y_2(\alpha, +o) = x_1(\alpha, +o) f_1'(\alpha),\\ &y_3(\alpha, +o) = \ldots\ldots \quad , \quad (|\alpha| \leqq 1).\end{aligned} \tag{3.9}$$

Analogous results are obtained for $y_i(\alpha, -o)$ on $\beta = -o$, $|\alpha| \leqq 1$. ($i = 1, 2, 3, \ldots$) Condition 3.8 prescribes the behaviour of the analytic functions $z_i(\eta)$ at infinity:

$$z_i(\eta) \sim a_i\eta \quad \text{for } \alpha^2 + \beta^2 \to \infty \qquad (i = 1, 2, 3, \ldots)$$

At $\eta = \pm 1$ must hold: $z_i(\eta) = o$ ($i = 1, 2, 3, \ldots$), because of the mapping of the body edges $z = \pm 1$ on the points $\eta = \pm 1$.

After having determined the terms of the series expansion of $z(\eta)$ we can determine the terms of the series expansion of $\chi(\eta)$. It is possible, however, to formulate one boundary value problem for $\chi(\eta)$ at once:

$$\begin{aligned}&\text{Im}\,\chi = o \quad \text{on} \quad \beta = \pm o, \ |\alpha| \leqq 1,\\ &\chi(\eta) \sim \chi_H\{z(\eta)\} \sim Ue^{-i\gamma}\,\eta(1 + a_1\epsilon + \ldots) \text{ at infinity},\\ &\chi(\eta) \text{ finite on } \beta = \pm o, \ |\alpha| \leqq 1.\end{aligned} \tag{3.10}$$

4. *Determination of* $\chi(\eta)$ *and* $z_1(\eta)$

We have the following boundary value problem for the sectionally holomorphic function $z_1(\eta)$:

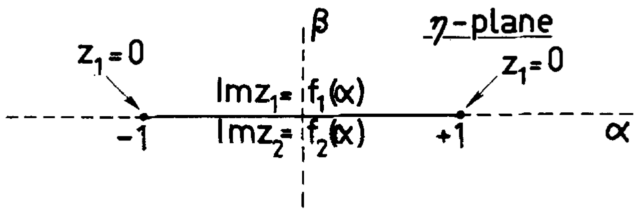


Fig. 2.

$$\beta=+0,\quad -1\leqq\alpha\leqq 1\quad :\quad \operatorname{Im} z_1=f_1(\alpha)=(1-\alpha^2)^{\frac{1}{2}}F_1(\alpha)\ ,$$
$$\beta=-0,\quad -1\leqq\alpha\leqq 1\quad :\quad \operatorname{Im} z_1=f_2(\alpha)=(1-\alpha^2)^{\frac{1}{2}}F_2(\alpha)\ ,$$

Condition at infinity: $z_1(\eta)=O(\eta)$,

Additional Condition: $z_1(\pm 1)=0.$ (4.1)

(See also Fig. 2)

This problem is a special case of a slightly more general boundary value problem treated in the appendix of this paper. According to expression A.8 from this appendix, the solution of problem 4.1 is given by

$$z_1(\eta)=A\eta+B+\frac{1}{2\pi i}(\eta^2-1)^{\frac{1}{2}}\int_{-1}^{+1}\frac{\{f_1(t)+f_2(t)\}\,dt}{(1-t^2)^{\frac{1}{2}}(t-\eta)}+$$
$$+\frac{1}{2\pi}\int_{-1}^{+1}\frac{\{f_1(t)-f_2(t)\}\,dt}{t-\eta}\,. \tag{4.2}$$

Because of the condition at infinity the homogeneous part of the solution has been reduced to $A\eta+B$. The real constants A and B can be determined by means of the additional conditions:

$$z_1(+1)=\frac{1}{2\pi}\int_{-1}^{+1}\frac{f_1(t)-f_2(t)}{t-1}\,dt+A+B=0,$$

$$z_1(-1)=\frac{1}{2\pi}\int_{-1}^{+1}\frac{f_1(t)-f_2(t)}{t+1}\,dt-A+B=0,$$

yielding

$$A=-\frac{1}{2\pi}\int_{-1}^{+1}\frac{f_1(t)-f_2(t)}{t^2-1}\,dt\quad\text{and}\quad B=-\frac{1}{2\pi}\int_{-1}^{+1}\frac{t\{f_1(t)-f_2(t)\}}{t^2-1}\,dt. \tag{4.3}$$

Substitution of 4.3 into 4.2 and rearranging gives:

$$z_1(\eta)=\frac{1}{2\pi i}(\eta^2-1)^{\frac{1}{2}}\int_{-1}^{+1}\frac{\{f_1(t)+f_2(t)\}\,dt}{(1-t^2)^{\frac{1}{2}}(t-\eta)}+\frac{1}{2\pi}(1-\eta^2)\int_{-1}^{+1}\frac{\{f_1(t)-f_2(t)\}\,dt}{(1-t^2)(t-\eta)}. \tag{4.4}$$

The behaviour of $z_1(\eta)$ at infinity is

$$z_1(\eta) \sim a_1\eta, \text{ with } a_1=A \text{ given by 4.3.} \tag{4.5}$$

We have now determined $z_1(\eta)$ and $x_1(\alpha,\pm o)$ can be computed from it in order to formulate boundary condition 3.9 for the next perturbation term $z_2(\eta)$. The resulting boundary value problem will not be solved here.
The solution of the homogeneous boundary value problem 3.10 for $\chi(\eta)$ can be given by means of expression A.8 from the appendix:

$$\chi(\eta)=D\eta+E+iF\sqrt{\eta^2-1}.$$

The real constants D, E and F follow from the condition at infinity

$$\chi(\eta) \sim Ue^{-i\gamma}\,\eta a(\epsilon) \text{ for } \alpha^2+\beta^2 \to \infty.$$

Thus we find

$$\chi(\eta)=(U\,\eta\cos\gamma - iU\sin\gamma\,.\,\sqrt{\eta^2-1})\,a(\epsilon), \tag{4.6}$$

or

$$\begin{aligned}\chi(\eta)&=\chi_o(\eta)+\epsilon\chi_1(\eta)+\epsilon^2\chi_2(\eta)+\ldots=\\&=(U\,\eta\cos\gamma - iU\sin\gamma\sqrt{\eta^2-1})(1+a_1\epsilon+a_2\epsilon^2+\ldots).\end{aligned} \tag{4.7}$$

5. *The uniform first order approximation*

In general we are only interested in a first order approximation of the problem, so we consider

$$\chi(\eta)=\chi_o(\eta)+\epsilon\chi_1(\eta)+R(\eta)$$
$$z(\eta)=\eta+\epsilon z_1(\eta)+Z(\eta)$$

The remainder term $Z(\eta)=X(\alpha,\beta)+iY(\alpha,\beta)$ is an analytic function of η in the cut η-plane and is of a degree not higher than one at infinity. Furthermore $Z(\eta)$ vanishes at $\eta=\pm 1$. Hence the real and imaginary part of $Z(\eta)$ are bounded functions of α on the upper and lower side of the line $\beta=o$, $|\alpha|\leqq 1$, and zero at the ends $\eta=\pm 1$.
So $Z(\eta)$ can be expressed in a way similar to $z_1(\eta)$:

$$Z(\eta)=\frac{1}{2\pi i}(\eta^2-1)^{\frac{1}{2}}\int_{-1}^{+1}\frac{(Y^++Y^-)dt}{(1-t^2)^{\frac{1}{2}}(t-\eta)}+\frac{1}{2\pi}\int_{-1}^{+1}\frac{(Y^+-Y^-)dt}{(1-t^2)(t-\eta)}\,. \tag{5.1}$$

Applying the mean value theorem to 3.7 and taking into account the boundary values 4.1 of $z_1(\eta)$, we find on $\beta=+o$, $|\alpha|\leqq 1$:

$$Y^+(\alpha)=Y(\alpha,+o)=\{\epsilon^2 x_1(\alpha,+o)+\epsilon X(\alpha,+o)\}\,f_1'\{\theta(\alpha)\},\ \text{with } |\theta(\alpha)|<1. \tag{5.2}$$

From 5.2 follows that $Y^+(\alpha)$ is of order of magnitude ϵ on $|\alpha|\leqq 1$. The same holds for $Y^-(\alpha)$. Then from 5.1 can be deduced that both $X(\alpha,+o)$ and $X(\alpha,-o)$ are of order of magnitude ϵ on $|\alpha|\leqq 1$. Then from 5.2 again follows that $Y^+(\alpha)$ is of order of magnitude ϵ^2, in other words

$$|Y^+(\alpha)|<\epsilon^2K_1,\quad |Y^-(\alpha)|<\epsilon^2K_2 \quad \text{on } |\alpha|\leqq 1.$$

(K_1 and K_2 real constants).

Hence $Z(\eta)$ is of order of magnitude ϵ^2 for finite η, and is $0(\epsilon^2\eta)$ at infinity.

The behaviour of $Z(\eta)$ near the ends $\eta=\pm 1$ can be found as follows:

It is always possible to map the exterior of the contours of the bodies given by 2.3 conformally on the exterior of the unit circle $|\zeta|=1$ in a ζ-plane by means of an analytic function $z=\tilde{z}(\zeta)$. This function is unique if we require correspondance of the points $z=\pm 1$ with the points $\zeta=\pm 1$, and if we require that $\tilde{z}(\zeta)=0(\zeta)$ at infinity.

So we have $\tilde{z}(\zeta)=\pm 1$ at $\zeta=\pm 1$, (5.3)

and, because of the conformity

$$\tilde{z}'(\zeta)\neq 0 \text{ at } \zeta=\pm 1. \tag{5.4}$$

By means of the transformation

$$\eta=\tfrac{1}{2}(\zeta+\frac{1}{\zeta});\ (\zeta=\eta+\sqrt{\eta^2-1})$$

the cut η-plane is mapped onto the exterior of the unit circle $|\zeta|=1$ in the ζ-plane.
So

$$\tilde{z}(\zeta)=z\{\eta(\zeta)\}=\tfrac{1}{2}(\zeta+\frac{1}{\zeta})+\epsilon z_1\{\eta(\zeta)\}+Z\{\eta(\zeta)\}.$$

Because of 5.3 and 5.4 we have

$$\epsilon z_1\{\eta(\zeta)\}+Z\{\eta(\zeta)\}=0(\zeta^2-1) \text{ at } \zeta=\pm 1,$$

or

$$\epsilon z_1(\eta)+Z(\eta)=0(\sqrt{\eta^2-1}) \text{ at } \eta=\pm 1.$$

From eq. 4.4 can be deduced

$$z_1(\eta)=0(\sqrt{\eta^2-1}) \text{ at } \eta=\pm 1.$$

So

$$Z(\eta)=0(\sqrt{\eta^2-1}) \text{ or } Z(\eta)=o(\sqrt{\eta^2-1}) \text{ at } \eta=\pm 1.$$

Resuming, we can say that for all η both $Z(\eta)$ and $Z'(\eta)$ give a contribution to $z(\eta)$ and $z'(\eta)$ respectively, that is of order ϵ compared to that of $\epsilon z_1(\eta)$ and $\epsilon z_1'(\eta)$ respectively. Hence $\eta+\epsilon z_1(\eta)$ is a uniform asymptotic approximation of $z(\eta)$.

The uniformity of the asymptotic approximation of $\chi(\eta)$ by $\chi_0(\eta)+\epsilon\chi_1(\eta)$ is clear from eq. 4.7. The complex velocity $w(\eta)$ is now approximated uniformly by

$$w=u-iv=\frac{\chi'(\eta)}{z'(\eta)}\approx\frac{\chi_0'(\eta)+\epsilon\chi_1'(\eta)}{1+\epsilon z_1'(\eta)}.$$

6. Examples. I) $F_1(x)$ and $F_2(x)$ polynomials in x.

If $F_1(x)$ and $F_2(x)$ are given polynomials in x, we can write

$$F_1(x)-F_2(x)=\sum_{k=0}^{N} q_k x^k; \tag{6.1}$$

$$F_1(x)+F_2(x)=(1-x^2)\sum_{k=0}^{N-2} p_k x^k, \tag{6.2}$$

where N is the highest degree of x in $F_1(x)$ and $F_2(x)$.
The factor $(1-x^2)$ in 6.2 occurs because of the condition

$$F_1(x)=-F_2(x)\neq 0 \text{ at } x=\pm 1.$$

Substitution of 6.1 and 6.2 into $z_1(\eta)$ gives

$$z_1(\eta)=\tfrac{1}{2}(\eta^2-1)^{\frac{1}{2}}\{F_1(\eta)-F_2(\eta)\}-\frac{i}{2\pi}\{F_1(\eta)+F_2(\eta)\}(\eta^2-1)^{\frac{1}{2}}\ln\frac{\eta-1}{\eta+1}+$$
$$+i(\eta^2-1)^{\frac{1}{2}}R^{(1)}_{N-1}(\eta)+(1-\eta^2)R^{(2)}_{N-1}(\eta), \tag{6.3}$$

with $R^{(1)}_{N-1}(\eta)$ and $R^{(2)}_{N-1}(\eta)$ polynomials of degree N-1 in η with real coefficients.

This solution of problem 4.1 for $z_1(\eta)$ can be verified directly:
The terms containing F_1-F_2 and F_1+F_2 have imaginary parts

$$\tfrac{1}{2}\{F_1(\alpha)-F_2(\alpha)\}(1-\alpha^2)^{\frac{1}{2}} \text{ and } \tfrac{1}{2}\{F_1(\alpha)+F_2(\alpha)\}(1-\alpha^2)^{\frac{1}{2}}$$

respectively, on the upper side of the cut, and have imaginary parts

$$-\tfrac{1}{2}\{F_1(\alpha)-F_2(\alpha)\}(1-\alpha^2)^{\frac{1}{2}} \text{ and } \tfrac{1}{2}\{F_1(\alpha)+F_2(\alpha)\}(1-\alpha^2)^{\frac{1}{2}}$$

respectively, on the lower side of the cut. So it is seen that the sum of these two terms is a particular solution of problem 4.1.

The last two terms of eq. 6.3 are homogeneous solutions of problem 4.1 in which the real coefficients of $R^{(1)}_{N-1}(\eta)$ and $R^{(2)}_{N-1}(\eta)$ are determined in such a way, that the degree of $z_1(\eta)$ at infinity is not higher than one.

II. Symmetric bodies.

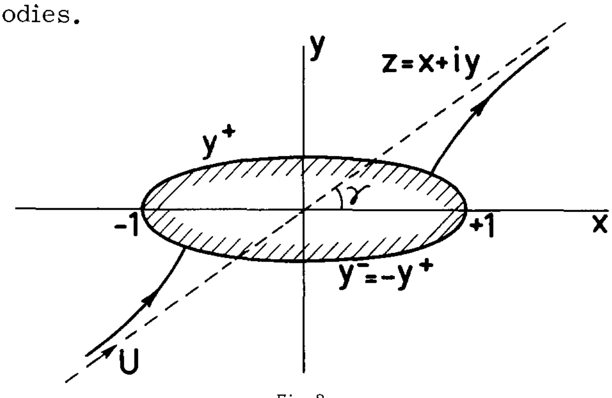


Fig. 3.

When the airfoil is symmetric with respect to the x-axis (see fig. 3), we have

$$F_1(x)+F_2(x)=0 \quad\text{and}\quad F_1(x)=-F_2(x)=F(x).$$

Taking

$$F(x)=\sum_{k=0}^{N} A_k x^k,$$

eq. 6.3 reduces to

$$z_1(\eta)=F(\eta)(\eta^2-1)^{\frac{1}{2}}+(1-\eta^2)R_{N-1}(\eta).$$

So we have the first order approximation

$$\left.\begin{aligned}\chi(\eta)&=\left[U\eta\cos\gamma-iU(\eta^2-1)^{\frac{1}{2}}\sin\gamma\right](1+a_1\epsilon)\\ z(\eta)&=\eta+\epsilon\left[F(\eta)(\eta^2-1)^{\frac{1}{2}}+(1-\eta^2)R_{N-1}(\eta)\right]\end{aligned}\right\}\qquad(6.4)$$

The complex velocity w=u-iv is then given by

$$w(\eta)=\frac{\left[U\cos\gamma-iU\eta(\eta^2-1)^{-\frac{1}{2}}\sin\gamma\right](1+a_1\epsilon)}{1+\epsilon\left[\eta(\eta^2-1)^{-\frac{1}{2}}F(\eta)+F'(\eta)(\eta^2-1)^{\frac{1}{2}}+(1-\eta^2)R'_{N-1}(\eta)-2\eta R_{N-1}(\eta)\right]}.\qquad(6.5)$$

The velocities at the ends $\eta=\pm1$ are

$$w(\pm1)=\frac{-iU\sin\gamma(1+a_1\epsilon)}{\epsilon F(\pm1)}.$$

The stagnation points are given by $\chi'(\eta)=0$, or

$$\eta=\cos\gamma \text{ on } \beta=+0, \text{ and } \eta=-\cos\gamma \text{ on } \beta=-0.$$

When the airfoil is an ellipse, we have F(x)=1, and in order to satisfy the condition at infinity:

$$z_1(\eta)=0(\eta) \text{ for } \alpha^2+\beta^2\to\infty,$$

the polynomial $R_{N-1}(\eta)$ must vanish identically. Thus we obtain the solution for the elliptic cylinder:

$$\left.\begin{aligned}\chi(\eta)&=\left[U\eta\cos\gamma-iU(\eta^2-1)^{\frac{1}{2}}\sin\gamma\right](1+\epsilon),\\ z(\eta)&=\eta+\epsilon(\eta^2-1)^{\frac{1}{2}},\end{aligned}\right\}$$

which is, in fact, the exact solution. (C. Jacob [5]).

III. Sharp edges and other body shapes.

In our theory the singular behaviour of $\chi'(\eta)$ at $\eta=\pm1$ does not depend on the body shape:

$$\chi'(\eta)=O\{\sin\gamma\,.\,(\eta^2-1)^{\frac{1}{2}}\} \text{ at } \eta=\pm1.$$

The nature of the singularities of $z(\eta)$, however, is strongly influenced by the shape of the leading and trailing edge of the body. In the case of elliptic ends the singularities of $z'(\eta)$ are the same as that of $\chi'(\eta)$, except for $\gamma=0$, thus yielding finite velocities on the airfoil.

In the case of other blunt symmetrical end-shapes, for instance

$$y(x)=O\{(1+x)^{\lambda}\} \text{ at } x=-1,\quad 0<\lambda<1,$$

we still can compute $z_1(\eta)$ from eq. 4.4. According to Mushkelishvili [6], the behaviour of $z_1(\eta)$ at $\eta=-1$ is then

$$z_1(\eta)=O\{(\eta+1)^{\lambda}\}.$$

This means that for $0<\lambda<\frac{1}{2}$ the velocity at the leading edge vanishes at any angle of incidence, and for $\frac{1}{2}<\lambda<1$ always is infinite (except for $\gamma=0$). This of course is physically intolerable.

The reason of this failure is that in these cases the expression $\eta+\epsilon z_1(\eta)$, as obtained from eq. 4.4, is not a uniformly valid approximation of the conformal mapping $z=z(\eta)$ of the airfoil on the cut $\beta=o$, $|\alpha|\leqq 1$ in the η-plane. Indeed, the singularity of $z_2(\eta)$ becomes $O\{(\eta+1)^{2\lambda-1}\}$ at $\eta=-1$, which makes correspondancy of $z=-1$ to $\eta=-1$ impossible for $0<\lambda<\frac{1}{2}$. For $\frac{1}{2}<\lambda<1$, the singularity in $z_3(\eta)$ would cause the same difficulty.

If the airfoil has a symmetrical wedge-shaped edge, i.e.

$$y(x)=\pm O(1-x) \quad \text{at} \quad x=1,$$

the results can be shown to be asymptotically correct.

In the case of a simple, symmetrical lenticular airfoil given by

$$y(x)=\pm\,\epsilon(1-x^2) \quad \text{for} \quad |x|\leqq 1,$$

eq. 4.4 yields

$$z(\eta)=\eta+\epsilon z_1(\eta)=\eta+\frac{\epsilon}{\pi}\,(1-\eta^2)\,\ln\frac{\eta-1}{\eta+1} \tag{6.6}$$

It is easy to show by comparison to the exact singularity

$$z(\eta)=O\left\{(\eta-1)^{1-\frac{1}{\pi}\operatorname{arctg} 2\epsilon}\right\} \quad \text{at} \quad \eta=1$$

that eq. 6.6 is a uniformly valid approximation of $z(\eta)$. In the same way we can show that $\{z'(\eta)\}^{-1}$, which we need for the computation of the velocities, is asymptotically correct. Of course we get infinite velocities at the sharp edges if the angle of incidence is unequal to zero.

Thus it is possible to give a uniformly valid approximation of the flow along an airfoil with blunt leading edge and wedge-shaped trailing edge. Let the airfoil be given by

$$y^+(x)=\epsilon(1-x)\sqrt{1+x}\;F_1(x)\,,$$

$$y^-(x)=\epsilon(1-x)\sqrt{1+x}\;F_2(x)\,,\quad |x|\leqq 1,$$

with $F_1(x)$ and $F_2(x)$ continuous, differentiable functions of x satisfying $F_1(\pm 1)=-F_2(\pm 1)\neq 0$.

The singular behaviour of $z_1(\eta)$ becomes

$$z_1(\eta)=O\{(\eta+1)^{\frac{1}{2}}\} \quad \text{at} \quad \eta=-1\,,$$

$$z_1(\eta)=O\{(\eta-1)\,\ln\,(\eta-1)\} \quad \text{at} \quad \eta=1\,,$$

yielding infinite velocities at the sharp trailing edge. We now can apply the well-known Kutta-condition in order to obtain finite velocities at the trailing edge by adding a circulation of strength Γ to $\chi(\eta)$:

$$\chi(\eta)=\{U\eta\cos\gamma-iU\sin\gamma\,(\eta^2-1)^{\frac{1}{2}}\}\,(1+a_1\epsilon)+\frac{i\Gamma}{2\pi}\ln\{\eta+(\eta^2-1)^{\frac{1}{2}}\}.$$

Γ is determined by requiring that $\eta=1$ be a stagnation point, or, what is the same, by requiring $\chi'(\eta)$ to be finite at $\eta=1$:

$$\Gamma=2\pi U\sin\gamma\,.\,(1+a_1\epsilon).$$

The complete uniformly valid first order solution is then

$$\chi(\eta)=\left\{U\eta\cos\gamma - iU\sin\gamma\left[(\eta^2-1)^{\frac{1}{2}}-\ln\left\{\eta+(\eta^2-1)^{\frac{1}{2}}\right\}\right]\right\}(1+a_1\epsilon),$$

$$z(\eta)=\eta+\frac{\epsilon}{2\pi i}(\eta-1)^{\frac{1}{2}}\int_{-1}^{+1}\frac{(1-t)^{\frac{1}{2}}[F_1(t)+F_2(t)]\,dt}{t-\eta}+\frac{\epsilon}{2\pi}(1-\eta^2)\int_{-1}^{+1}\frac{[F_1(t)-F_2(t)]\,dt}{(1+t)^{\frac{1}{2}}(t-\eta)},$$

with $$a_1=\frac{1}{2\pi}\int_{-1}^{+1}\frac{[F_1(t)-F_2(t)]\,dt}{(1+t)^{\frac{1}{2}}}.$$

7. *An alternative approach to the problem*

The uniformity of the approximation is based on the use of a new complex variable $\eta=\alpha+i\beta$ on which depend both $\chi=\Phi+i\Psi$ and $z=x+iy$. Formulating the problem in terms of the real variables α and β, we have the basic equations

$$\Phi_x=\Psi_y \quad \text{and} \quad \Phi_y=-\Psi_x \tag{7.1}$$

of which we want to give a solution of the form:

$$\Phi=\Phi(\alpha,\beta),\ \Psi=\Psi(\alpha,\beta),\ x=x(\alpha,\beta),\ y=y(\alpha,\beta).$$

We are free to choose either a set of two first order partial differential equations for $x(\alpha,\beta)$ and $y(\alpha,\beta)$ or a set of two first order equations for $\Phi(\alpha,\beta)$ and $\Psi(\alpha,\beta)$, because, having chosen a set of equations for $x(\alpha,\beta)$ and $y(\alpha,\beta)$, then from 7.1 follows a set of equations for $\Phi(\alpha,\beta)$ and $\Psi(\alpha,\beta)$, and reversely, having chosen a set of equations for $\Phi(\alpha,\beta)$ and $\Psi(\alpha,\beta)$, then also from 7.1 follows a set of equations for $x(\alpha,\beta)$ and $y(\alpha,\beta)$.

We have taken z as an analytic function of η, which is equivalent to choosing the equations

$$x_\alpha=y_\beta\ ;\ x_\beta=-y_\alpha \tag{7.2}$$

for $x(\alpha,\beta)$ and $y(\alpha,\beta)$.
Then from 7.1 follow the equations

$$\Phi_\alpha=\Psi_\beta \quad \text{and} \quad \Phi_\beta=-\Psi_\alpha, \tag{7.3}$$

which is equivalent to $x=\Phi+i\Psi$ being an analytic function of η.

The equations 7.2 and 7.3 can be obtained by a different way of reasoning: When the angle of incidence γ is equal to zero, the line $\Psi=o$ of the Φ,Ψ-plane corresponds to the line $\beta=o$ of the α,β-plane, in which we want the thin body to be represented by $\beta=o, |\alpha|\leqq 1$.

Moreover, we want the points of infinity of the α,β- and the x, y-plane to correspond as follows·

$$x \sim a(\epsilon)\alpha$$
$$y \sim b(\epsilon)\beta \quad \text{for } \alpha^2+\beta^2\to\infty,\ \text{(a and b real).}$$

From the x, y-plane we know

$$\Phi \sim Ux,\ \Psi \sim Uy \quad \text{for} \quad x^2+y^2\to\infty$$

So

$$\Phi \sim Ua(\epsilon)\alpha,\ \Psi \sim Ub(\epsilon)\beta, \quad \text{for } \alpha^2+\beta^2\to\infty.$$

Because of this resemblance of the Φ,Ψ-plane and the α,β-plane, it is reasonable to choose

$$\Phi(\alpha,\beta)=U\alpha a(\epsilon),$$
$$\Psi(\alpha,\beta)=U\beta b(\epsilon), \text{ for all } \alpha \text{ and } \beta. \tag{7.4}$$

Then the equations that must be satisfied by $x(\alpha,\beta)$ and $y(\alpha,\beta)$ follow from 7.1:

$$a(\epsilon)y_\beta=b(\epsilon)x_\alpha \quad \text{and} \quad a(\epsilon)x_\beta=-b(\epsilon)y_\alpha. \tag{7.5}$$

If we can determine $x(\alpha,\beta)$ and $y(\alpha,\beta)$ in such a way, that $a(\epsilon)=b(\epsilon)$ then we obtain equations 7.2 for $x(\alpha,\beta)$ and $y(\alpha,\beta)$.

Eqs. 7.2 imply that $z=x+iy$ is an analytic function of $\eta=\alpha+i\beta$, so if we require:

$$z(\eta) \sim C(\epsilon)\eta \quad \text{for} \quad \alpha^2+\beta^2 \to \infty, \; C(\epsilon) \text{ real},$$

then indeed the condition $a(\epsilon)=b(\epsilon)$ is satisfied.

The equations 7.2 are independent of the angle of incidence γ and also the boundary values of it only depend on the geometry of the body. Thus in the case $\gamma\neq 0$, when we can not choose $\Phi(\alpha,\beta)$ and $\Psi(\alpha,\beta)$ according to 7.4, we take eqs. 7.2 to be satisfied by $x(\alpha,\beta)$ and $y(\alpha,\beta)$. Then from 7.1 follows the set of equations 7.3, that have to be satisfied by $\Phi(\alpha,\beta)$ and $\Psi(\alpha,\beta)$.

This way of reasoning in order to obtain a set of equations for the four unknown functions $x(\alpha,\beta)$, $y(\alpha,\beta)$, $\Phi(\alpha,\beta)$ and $\Psi(\alpha,\beta)$ is entirely independent of the idea of conformal mapping, and can be useful in the case of more general elliptic systems of basic equations.

It is possible, however, that difficulties arise. For instance, when the basic equations have the non-linear form

$$\begin{aligned}\Phi_x&=A(\Phi_x,\Phi_y,\Psi_x,\Psi_y)\Psi_y\\ \Phi_y&=B(\Phi_x,\Phi_y,\Psi_x,\Psi_y)\Psi_x,\end{aligned} \tag{7.6}$$

substitution of eqs. 7.4 in the case of $\gamma=0$ yields, in general, a set of linear equations for $x_1(\alpha,\beta)$ and $y_1(\alpha,\beta)$, and non-linear equations for the higher order terms of $x(\alpha,\beta)$ and $y(\alpha,\beta)$. This is no disadvantage because it is still possible to give a first order approximative solution in the case $\gamma=o$.

But if we want to extend the method to $\gamma\neq o$, we have to put:

$$\begin{aligned}\Phi(\alpha,\beta)&=\Phi_o(\alpha,\beta)+\epsilon\Phi_1(\alpha,\beta)+\dots\\ \Psi(\alpha,\beta)&=\Psi_o(\alpha,\beta)+\epsilon\Psi_1(\alpha,\beta)+\dots\end{aligned} \tag{7.7}$$

Substitution of eqs. 7.7 into 7.6 and using the linear equations obtained for $x_1(\alpha,\beta)$ and $y_1(\alpha,\beta)$, produces non-linear equations even for Φ_o and Ψ_o, that are mostly unsolvable. This is the reason why in the case of compressible, subsonic flow the treatment will be restricted to $\gamma=o$. (See next section).

8. Two-dimensional subsonic compressible flow at zero angle of incidence

A two-dimensional compressible flow is governed by the equations

$$u=\Phi_x=\frac{\rho_o}{\rho(x,y)}\Psi_y; \qquad v=\Phi_y=-\frac{\rho_o}{\rho(x,y)}\Psi_x \tag{8.1}$$

(ρ density; ρ_o undisturbed density at infinity).
We consider ideal fluids obeying the pressure-density relation of Poisson

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^k, \qquad \left(k = \frac{C_p}{C_v}\right) \tag{8.2}$$

(p pressure; p_o undisturbed density at infinity; k specific heat ratio of the fluid)

Introducing the velocity of sound (a) by

$$a^2 = \frac{dp}{d\rho},$$

we can express $\frac{\rho_o}{\rho}$ as a function of a^2:

$$\frac{\rho_o}{\rho} = \left(\frac{a_o^2}{a^2}\right)^{\frac{1}{k-1}} \quad (a_o\text{: velocity of sound at infinity})$$

With the aid of Bernoulli's law

$$\tfrac{1}{2}(u^2+v^2) + \frac{a^2}{k-1} = \tfrac{1}{2}\,U^2 + \frac{a_o^2}{k-1}$$

we find two basic equations for our problem:

$$\Phi_x = \left\{1 + \frac{k-1}{2a_o^2}\,(U^2-\Phi_x^2-\Phi_y^2)\right\}^{\frac{1}{1-k}}\Psi_y \tag{8.3}$$

$$\Phi_y = -\left\{1 + \frac{k-1}{2a_o^2}\,(U^2-\Phi_x^2-\Phi_y^2)\right\}^{\frac{1}{1-k}}\Psi_x$$

We consider the same class of thin bodies as in the incompressible case, placed in a uniform compressible flow with velocity U and angle of incidence $\gamma=0$.

According to the approach pointed out in section 7, we put

$$\begin{aligned}\Phi(\alpha,\beta) &= U\alpha a(\epsilon) = U\alpha(1+a_1\epsilon+a_2\epsilon^2+\dots)\\ \Psi(\alpha,\beta) &= U\beta b(\epsilon) = U\beta(1+b_1\epsilon+b_2\epsilon^2+\dots)\end{aligned} \tag{8.4}$$

where $a(\epsilon)$ and $b(\epsilon)$ depend on the behaviour of $x(\alpha,\beta)$ and $y(\alpha,\beta)$ at infinity:

$$x(\alpha,\beta) \sim \alpha a(\epsilon),$$

$$y(\alpha,\beta) \sim \beta b(\epsilon) \quad \text{for} \quad \alpha^2+\beta^2 \to \infty.$$

Substitution of eqs. 8.4 and of

$$x(\alpha,\beta) = \alpha + \epsilon x_1(\alpha,\beta) + \dots$$

$$y(\alpha,\beta) = \beta + \epsilon y_1(\alpha,\beta) + \dots$$

into eqs. 8.3 and neglecting terms of higher order than ϵ^2, yields

$$\frac{\partial y_1}{\partial \beta} - (1-M_o^2)\frac{\partial x_1}{\partial \alpha} = b_1-(1-M_o^2)a_1,$$

$$\frac{\partial y_1}{\partial \alpha} + \frac{\partial x_1}{\partial \beta} = 0, \tag{8.5}$$

where $M_o=U/a_o$ is the Mach-number of the flow at infinity. We will consider subsonic flows, i.e. $M_o<1$.

If it is possible to determine $x_1(\alpha,\beta)$ and $y_1(\alpha,\beta)$ such that

$$b_1-(1-M_o^2)a_1=0 \tag{8.6}$$

then we have after putting $m^2=1-M_o^2$:

$$\frac{\partial y_1}{\partial \beta} - m^2\frac{\partial x_1}{\partial \alpha}=0;\quad \frac{\partial x_1}{\partial \beta}+\frac{\partial y_1}{\partial \alpha}=0 \tag{8.7}$$

These two equations are the Cauchy-Riemann equations for the function $x_1+\frac{i}{m}y_1$, that is analytic in the complex variable $\alpha+im\beta$. Requiring:

$$x_1+\frac{i}{m}y_1 \sim C(\alpha+im\beta) \text{ for } \alpha^2+\beta^2\to\infty \qquad (C \text{ real}),$$

then indeed condition 8.6 is satisfied.

It is easy to see now, that a compressible thin body flow at zero angle of attack can be derived from the corresponding incompressible flow by means of the following rule:

Let the first-order approximation of an incompressible flow of zero angle of incidence be given by

$$\chi(\eta)=U\eta(1+a_1\epsilon), \qquad (\eta=\alpha+i\beta)$$

$$z(\eta)=\eta+\epsilon z_1(\eta),$$

with

$$z(\eta)\sim(1+a_1\epsilon)\eta \qquad \text{for } \alpha^2+\beta^2\to\infty,$$

then the first-order approximation of the corresponding compressible subsonic flow is given by

$$\Phi(\alpha,\beta)+\frac{i}{m}\Psi(\alpha,\beta)=U(\alpha+i\frac{\beta}{m})+\frac{\epsilon U a_1}{m}(\alpha+i\beta m)$$

$$x(\alpha,\beta)+\frac{i}{m}y(\alpha,\beta)=\alpha+i\frac{\beta}{m}+\frac{\epsilon}{m}z_1(\alpha+i\beta m)$$

This rule can be considered as the analogy of the well-known Prandtl-Glauert rule from the classical subsonic airfoil theory.

APPENDIX: The solution of a certain boundary value problem

We are looking for a sectionally holomorphic function $W(z)$, being of finite degree at infinity, satisfying the following boundary conditions on the cut $y=o$, $|x|\leqq 1$ of the complex z-plane:

$$y=+o,\ |x|\leqq 1: \quad \operatorname{Im} W(z)=f_1(x) \tag{A.1}$$

$$y=-o,\ |x|\leqq 1: \quad \operatorname{Im} W(z)=f_2(x)$$

Additional condition: $y=o$, $|x|=1$: W finite.

The real functions $f_1(x)$ and $f_2(x)$ have the form

$$f_i(x)=(1-x)^{a_i}(1+x)^{b_i}F_i(x),$$

(with $a_i>0$, $b_i>0$, $F_i(x)$ continuous, $F_i(\pm 1)\neq 0$; $i=1,2$).

This boundary value problem is a modification of a problem treated by N. I. Muskhelishvili [6].

We put: $\Omega(z)=\frac{1}{2}\{W(z)+\overline{W(\bar{z})}\}$, and

$\chi(z)=\frac{1}{2}\{W(z)-\overline{W(\bar{z})}\}$,

from which follows:

$$\Omega(z)=\overline{\Omega(\bar{z})}, \text{ and} \tag{A.2}$$

$$\chi(z)=-\overline{\chi(\bar{z})} \tag{A.3}$$

On the real axis we have now:

$$\begin{aligned}
&\Omega(x+io)+\Omega(x-io)= \\
&=\Omega^++\Omega^-=2\operatorname{Re}\Omega^+=\operatorname{Re}(W^++W^-) \\
&\Omega^+-\Omega^-=2i\operatorname{Im}\Omega^+=i\operatorname{Im}(W^+-W^-) \\
&\chi^++\chi^-=2i\operatorname{Im}\chi^+=i\operatorname{Im}(W^++W^-) \\
&\chi^+-\chi^-=2\operatorname{Re}\chi^+=\operatorname{Re}(W^+-W^-)
\end{aligned}$$

The boundary value problem A.1 can be separated into two Hilbert problems:

$$\begin{aligned}
y=o,\quad -\infty<x<-1 &: \quad \Omega^+-\Omega^-=0 \\
-1\leqq x\leqq +1 &: \quad \Omega^+-\Omega^-=i[f_1(x)-f_2(x)] \\
+1<x<\infty &: \quad \Omega^+-\Omega^-=0
\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}
y=o,\quad -\infty<x<-1 &: \quad \chi^+-\chi^-=0 \\
-1\leqq x\leqq +1 &: \quad \chi^++\chi^-=i[f_1(x)+f_2(x)] \\
+1<x<\infty &: \quad \chi^+-\chi^-=0
\end{aligned} \tag{A.5}$$

A particular solution of problem A.4 is

$$\Omega(z)=\frac{1}{2\pi}\int_{-1}^{+1}[f_1(t)-f_2(t)]\,\frac{dt}{t-z} \tag{A.6}$$

The general solution of an inhomogeneous boundary value problem is found by adding a particular solution of it to the general solution of the homogeneous problem. So the solution of problem A.4, satisfying A.2, being finite at $z=\pm 1$, and being of finite degree at infinity is:

$$\Omega(z)=\frac{1}{2\pi}\int_{-1}^{+1}[f_1(t)-f_2(t)]\frac{dt}{t-z}+R_1(z) \tag{A.7}$$

where $R_1(z)$ is a polynomial of finite degree in z, with real coefficients.

A particular solution of problem A.5 is constructed from a particular solution χ_p of the corresponding homogeneous problem by means of

$$\chi(z)=\frac{1}{2\pi i}\chi_p(z)\int_{-1}^{+1}\frac{i[f_1(t)+f_2(t)]\,dt}{\chi_p^+(t)\,(t-z)}.$$

We take $\chi_p(z)=i(z^2-1)^{\frac{1}{2}}$, choosing that branch of $(z^2-1)^{\frac{1}{2}}$ for which holds

$$(z^2-1)^{\frac{1}{2}}\sim z \quad \text{for} \quad x^2+y^2\to\infty$$

The general solution of the homogeneous problem corresponding to A.5 is

$$\chi_H(z)=iR_2(z)(z^2-1)^{\frac{1}{2}},$$

where $R_2(z)$ is a polynomial of finite degree in z with real coefficients. The general solution of problem A.5 is then:

$$\chi(z)=-\frac{1}{2\pi}(z^2-1)^{\frac{1}{2}}\int_{-1}^{+1}\frac{i[f_1(t)+f_2(t)]\,dt}{(1-t^2)^{\frac{1}{2}}(t-z)}+iR_2(z)(z^2-1)^{\frac{1}{2}}.$$

This solution is of finite degree at infinity, is finite for $z=\pm 1$ and satisfies A.3.

The solution W(z) of the original boundary value problem A.1 is given by

$$W(z)=\Omega(z)+\chi(z)=R_1(z)+iR_2(z)(z^2-1)^{\frac{1}{2}}+$$

$$+\frac{1}{2\pi i}(z^2-1)^{\frac{1}{2}}\int_{-1}^{+1}\frac{[f_1(t)+f_2(t)]\,dt}{(1-t^2)^{\frac{1}{2}}(t-z)}+\frac{1}{2\pi}\int_{-1}^{+1}\frac{[f_1(t)-f_2(t)]\,dt}{t-z}.$$

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